Hale School

Mathematics Specialist

Test 3 --- Term 2 2017

Vectors in 3D

Name: SOLUTIONS

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Instructions:

- **CAS calculators are allowed**
- **External notes are not allowed**
- **Duration of test: 45 minutes**
- **Show your working clearly**
- **Use the method specified (if any) in the question to show your working (Otherwise, no marks awarded)**
- **This test contributes to 7% of the year (school) mark**

Question 1 **(8 marks: 2, 2, 2 and 2)**

Consider points A (1 , 3 , 5) , B (-7 , 5 , 1) and C (3 , -2 , 4).

(a) Determine the vectors **AB** and **AC**.

$\vec{AB} = \begin{pmatrix} -8 \\ +2 \\ -4 \end{pmatrix}$ ✓ $\vec{AB}$

$\vec{AC} = \begin{pmatrix} 2 \\ -5 \\ -1 \end{pmatrix}$ ✓ $\vec{AC}$

(b) Determine the vector equation of line containing points A and B.

$$\underset{\sim}{r} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 1 \\ -2 \end{pmatrix}$$

✓ uses $\underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{m}$

✓ all correct

(c) Find a vector perpendicular to both **AB** and **AC**.

$$\begin{pmatrix} -4 \\ 1 \\ -2 \end{pmatrix} \times \begin{pmatrix} 2 \\ -5 \\ -1 \end{pmatrix} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ -4 & +1 & -2 \\ 2 & -5 & -1 \end{vmatrix}$$

$$= -11\underset{\sim}{i} - 8\underset{\sim}{j} + 18\underset{\sim}{k}$$

✓ uses cross product

Perpendicular vector is

$$11\underset{\sim}{i} + 8\underset{\sim}{j} - 18\underset{\sim}{k}$$

✓ correct answer

(d) Find the Cartesian equation of the plane passing through A, B and C.

$$\underset{\sim}{r} \cdot \underset{\sim}{n} = \underset{\sim}{a} \cdot \underset{\sim}{n}$$

$$\underset{\sim}{r} \cdot \begin{pmatrix} 11 \\ 8 \\ -18 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 11 \\ 8 \\ -18 \end{pmatrix}$$

✓ uses $\underset{\sim}{r} \cdot \underset{\sim}{n} = \underset{\sim}{a} \cdot \underset{\sim}{n}$

$$11x + 8y - 18z = -55$$

✓ eq^n of plane (cartesian form)

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Question 2 **(8 marks: 3, 5)**

The vector equations of lines L and M are given by

$$\vec{r}_L = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \quad \text{and} \quad \vec{r}_M = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \quad \text{respectively.}$$

(a) Show that the lines do not intersect.

$x: \quad -2 + \lambda = 1 + 3\mu$

$y: \quad 1 - \lambda = 2 + \mu$

$-1 = 3 + 4\mu$

$\therefore \mu = -1$

$\lambda = 0$

✓ solves for μ, λ

Check z: $1 + 2\lambda = 1$

$-1 + 4\mu = -5$

✓ checks 3rd component

Not equal $\therefore$ no intersection

✓ conclusion

(b) If P and Q are points on L and M, the two lines are closest when PQ is perpendicular to both line L and line M. Find the closest distance between the two lines accurate to 2 decimal places.

$$\vec{PQ} = \begin{pmatrix} 1+3\mu \\ 2+\mu \\ -1+4\mu \end{pmatrix} - \begin{pmatrix} -2+\lambda \\ 1-\lambda \\ 1+2\lambda \end{pmatrix} = \begin{pmatrix} 3+3\mu-\lambda \\ 1+\mu+\lambda \\ -2+4\mu-2\lambda \end{pmatrix}$$

✓ $\vec{PQ}$

$\vec{PQ} \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = 0 \Rightarrow 3 + 3\mu - \lambda - 1 - \mu - \lambda - 4 + 8\mu - 4\lambda = 0$

$\Rightarrow 10\mu - 6\lambda = 2 \quad (1)$

✓ uses dot products

$\vec{PQ} \cdot \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = 0 \Rightarrow 9 + 9\mu - 3\lambda + 1 + \mu + \lambda - 8 + 16\mu - 8\lambda = 0$

$\Rightarrow 26\mu - 10\lambda = -2 \quad (2)$

Solving (1) and (2) simultaneously gives

✓ equations in μ, λ

$\mu = -\frac{4}{7}, \quad \lambda = -\frac{9}{7}$

Closest points are $\begin{pmatrix} -5/7 \\ 10/7 \\ -23/7 \end{pmatrix}$ and $\begin{pmatrix} -23/7 \\ 16/7 \\ -11/7 \end{pmatrix}$

✓ solves for μ, λ

Distance is $\left| \begin{pmatrix} 18/7 \\ -6/7 \\ -12/7 \end{pmatrix} \right| = \frac{6\sqrt{14}}{7} \simeq \underline{\underline{3.21}}$ (2dp)

✓ distance

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Question 3 **(5 marks: 3, 2)**

At 5am, hot-air balloons A and B leave their home bases located at $-10\mathbf{i}+40\mathbf{j}+0.2\mathbf{k}$ and $15\mathbf{i}+60\mathbf{j}+0.05\mathbf{k}$ with constant velocities $\mathbf{v}_A=10\mathbf{i}+40\mathbf{j}+\alpha\mathbf{k}$ and $\mathbf{v}_B=5\mathbf{i}+\beta\mathbf{j}+0.02\mathbf{k}$. Measurements are in km and km/hr.

(a) Find the values of α and β given that the two balloons collide.

$$\begin{pmatrix} -10+10t \\ 40+40t \\ 0.2+\alpha t \end{pmatrix} = \begin{pmatrix} 15+5t \\ 60+\beta t \\ 0.05+0.02t \end{pmatrix}$$ ✓ solves for t

x: $5t=25 \Rightarrow t=5$

y: $(40-\beta)t=20 \Rightarrow \beta=36$ ✓ solves for β

z: $0.15=(0.02-\alpha)t \Rightarrow \alpha=-0.01$ ✓ solves for α

(b) State the time and position of the collision.

✓ time

$t=5$ so collide at 10:00 am

Collide at $\begin{pmatrix} 40 \\ 240 \\ 0.15 \end{pmatrix}$ ✓ position

or $40\underline{i}+240\underline{j}+0.15\underline{k}$

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Question 4 **(8 marks: 2, 2, 2, 2)**

(a) In each case below, state whether the system of equations has a unique solution, no solutions or an infinite number of solutions. State the geometric relationship between the planes in each case.

i)
$$\begin{aligned} x + 2y + z &= 7 \\ 3x + 6y + 3z &= 11 \\ 2x - 3y - z &= 4 \end{aligned}$$

✓ No solution.

No solutions.

2 parallel planes and another intersecting

✓ Parallel plane

ii)
$$\begin{aligned} x + y - z &= 4 \\ 3x + 5y + 2z &= 7 \\ 2x + 4y + 3z &= 3 \end{aligned}$$

✓ ∞ solutions

Infinite solutions

✓ interpretation

The 3 planes intersect in a common line.

(b) Consider the system of equations in x, y and z with augmented matrix,

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 4 \\ 0 & 5 & -3 & 12 \\ 0 & 0 & p^2-4 & p-2 \end{array}\right]$$

i) Find the solution to the equations when p = -1

$-3z = -3 \Rightarrow z = 1$

✓ finds z

$5y - 3z = 12 \Rightarrow y = 3$

$2x - y + 3z = 4 \Rightarrow x = 2$

✓ finds y, x

ii) State the value(s) of p for which there are no solutions to the system of equations.

Needs $0z = a \quad a \neq 0$

✓ $0z = a$

$\therefore p^2 - 4 = 0$ and $p \neq 2$

$\therefore p = -2$

✓ $p = -2$

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Question 5 **(7 marks: 2, 5)**

Given the sphere with equation $\left| \vec{r} - \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \right| = 4$ and the plane with equation $x + 2y + z = 24$

(a) Find the equation of the straight line passing through the centre of the sphere that is perpendicular to the given plane.

$$\underset{\sim}{r} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

✓ uses $\underset{\sim}{r} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + \lambda \underset{\sim}{n}$

✓ $\underset{\sim}{n}$ correct

(b) Find the exact distance between the plane and the sphere.

Line meets plane when

$$(-1+\lambda) + 2(2+2\lambda) + (3+\lambda) = 24$$

$$\Rightarrow \quad 6\lambda = 18$$

$$\Rightarrow \quad \lambda = 3$$

Meet at $\begin{pmatrix} 2 \\ 8 \\ 6 \end{pmatrix}$

Distance is $\left| \begin{pmatrix} 2 \\ 8 \\ 6 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \right| - 4$

$$= \sqrt{3^2 + 6^2 + 3^2} - 4$$

$$= \sqrt{54} - 4$$

$$= 3\sqrt{6} - 4$$

✓ sets up eqⁿ

✓ $\lambda = 3$

✓ $\begin{pmatrix} 2 \\ 8 \\ 6 \end{pmatrix}$

✓ distance

✓ exact and simplified

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Question 6 **(11 marks: 3, 3, 3, 2)**

A particle moves along a path described by the vector function $\mathbf{r}(t) = (3 + 4\cos t)\mathbf{i} + 2\sin t\,\mathbf{j}$ for $t \geq 0$.

(a) Determine but do not simplify the Cartesian equation of the path.

$x = 3 + 4\cos t$ $\Rightarrow$ $\cos t = \frac{x-3}{4}$

$y = 2\sin t$ $\qquad$ $\sin t = y/2$

✓ rearranges

$\cos^2 t + \sin^2 t = 1$ $\qquad$ ✓ uses $\cos^2 t + \sin^2 t =$

$\Rightarrow \left(\frac{x-3}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$ $\qquad$ ✓ equation

(b) Show that the speed of the particle at time t is given by $\sqrt{4 + 12\sin^2 t}$.

$\text{Speed} = |\underline{v}(t)| = \left|\begin{pmatrix} -4\sin t \\ 2\cos t \end{pmatrix}\right|$ $\qquad$ ✓ $\underline{v}(t)$

$= \sqrt{16\sin^2 t + 4\cos^2 t}$ $\qquad$ ✓ speed formula

$= \sqrt{16\sin^2 t + 4 - 4\sin^2 t}$ $\qquad$ ✓ shows result

$= \sqrt{4 + 12\sin^2 t}$

(c) Determine the location(s) of the particle when it has minimum speed.

Min speed when $\sin t = 0$ $\qquad$ ✓ $\sin t = 0$

$\therefore t = 0, \pi, 2\pi, \ldots$ $\qquad$ ✓ $t = 0, \pi$

Location is $(3 \pm 4, 0)$

so $7\underline{i}$ or $-\underline{i}$ $\qquad$ ✓ both answers

(d) Write down and evaluate, to 2 decimal places, an integral that will determine the distance travelled by the particle in the first 5 seconds.

$\text{Distance} = \int_0^5 \sqrt{4 + 12\sin^2 t}\, dt$ $\qquad$ ✓ integral

$=$ ~~15.67 units~~ $\qquad$ ✓ value

10.04 units

(11)